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As an instance of this kind we may cite an article by Netto which appeared in *Crelle's Journal*, volume 128 (1905), page 243. We find in this article a development of the tetrahedral group according to an abstract definition which has been well known for more than half a century, and yet no references are given by the author of the article in question or by the reviewer in the *Fortschritte*. The article contains a number of other things which had been published several years before this article appeared. An instance of an article which contained nothing new and yet appeared in the excellent *Mathematische Annalen*, without proper reference, may be found by consulting this journal, volume 60 (1905), page 319.

Errors of Judgment.—There is a class of indirect errors to which it may be desirable to refer here; namely, those which arise from undue emphasis on some particular phase of the subject or on the work of some particular man. According to the writer's opinion Burnside's *Theory of Groups* errs along this line by quoting Hölder's work relatively too frequently. In fact, these references are not always correct. Even in the second edition, page 39, we are told that the symbol for quotient group was introduced by Hölder, although this symbol had been employed by Jordan at a much earlier date. As other instances of undue emphasis in important publications with respect to the work of one man, we may cite the references to Frobenius in the excellent article on groups contained in *Pascal's Repertorium* and the references to the present writer in the French encyclopedia article on this subject.

Conclusion.—It may be added that the above are only a few typical examples of the many errors which have been committed by some of those who have contributed much to the development of group theory. A complete collection of such errors, with proper explanations, would make a volume. Some of these errors have not had an unmitigatedly baneful effect, as they inspired young investigators with confidence, when these found that they could correct mistakes in the works of eminent mathematicians. The contributions to mathematics, due to the inspiration received from the discovery of errors in the works of eminent men, have been of great value to our science; but the beauty and insight furnished by correct and fundamental results are still more inspiring and are much more worthy objects of our endeavors.

ON THE REMAINDER TERM IN A CERTAIN DEVELOPMENT OF $f(a+x)$.*

By R. D. CARMICHAEL, Indiana University.

In the *Annals of Mathematics*, Vol. 5, No. 4, July, 1904, Mr. S. A. Corey gave an interesting and important development of $f(a+x)$, and showed that his formula was admirably adapted to the numerical computation of complicated

* Read before the American Mathematical Society, February 29, 1908.

definite integrals. The object of this paper is to find the "remainder term" in Corey's series* and incidentally to verify the series.

Putting the expansion in a form somewhat different from that of Corey and writing after it the remainder term R_{2n+2}, we shall verify the series and find the value† of R_{2n+2}:

$$
\begin{aligned}
f(a+x) &= f(a) + \frac{x}{m}\left[f'\left(a+\frac{x}{m}\right) + f'\left(a+\frac{2x}{m}\right) + \cdots + f'\left(a+\frac{m-1}{m}x\right)\right. \\
&\quad \left. + f'(a+x)\right] - \frac{x}{m\cdot 2}[f'(a+x)-f'(a)] - \frac{B_1x^2}{m^2\cdot 2!}[f''(a+x)-f''(a)] \\
&\quad + \frac{B_2x^4}{m^4\cdot 4!}[f^{(\mathrm{iv})}(a+x)-f^{(\mathrm{iv})}(a)] - \cdots \\
&\quad + (-1)^n\frac{B_nx^{2n}}{m^{2n}\cdot(2n)!}[f^{(2n)}(a+x)-f^{(2n)}(a)] + R_{2n+2}.
\end{aligned} \tag{1}
$$

Here m is a positive integer and $B_1, B_2, \cdots$ are the well-known Bernoulli numbers:

$$B_1 = \tfrac{1}{6}, \qquad B_2 = \tfrac{1}{30}, \qquad B_3 = \tfrac{1}{42}, \qquad B_4 = \tfrac{1}{30}, \qquad B_5 = \tfrac{5}{66}, \qquad \cdots.$$

Throughout the discussion it is assumed that $f(x)$ and its derivatives employed exist and are continuous over the range a to $a+x$.

When $r \leqq m$ one sees from Taylor's development with remainder term that

$$
\begin{aligned}
f'\left(a+\frac{rx}{m}\right) &= f'(a) + \frac{rx}{m}f''(a) + \frac{r^2x^2}{2!\cdot m^2}f'''(a) + \cdots \\
&\quad + \frac{r^{2n}x^{2n}}{m^{2n}\cdot(2n)!}f^{(2n+1)}(a) + \frac{r^{2n+1}x^{2n+1}}{m^{2n+1}\cdot(2n+1)!}f^{(2n+2)}(a+\varphi_r x),
\end{aligned} \tag{2}
$$

where $0 < \varphi_r < 1$. If in this equation r is put successively equal to $1, 2, 3, \cdots, m$, and the resulting equation in each case is multiplied by x/m, we have by addition:

$$
\begin{aligned}
&\frac{x}{m}\left[f'\left(a+\frac{x}{m}\right) + f'\left(a+\frac{2x}{m}\right) + \cdots + f'\left(a+\frac{m-1}{m}x\right) + f'(a+x)\right] \\
&\quad = xf'(a) + x^2f''(a)\left[\frac{1+2+3+\cdots+m}{m^2}\right] + \frac{x^3f'''(a)}{2!}\left[\frac{1^2+2^2+3^2+\cdots+m^2}{m^3}\right] \\
&\quad + \cdots + \frac{x^{2n+1}f^{(2n+1)}(a)}{(2n)!}\left[\frac{1^{2n}+2^{2n}+3^{2n}+\cdots+m^{2n}}{m^{2n+1}}\right] \\
&\quad + \frac{x^{2n+2}}{m^{2n+2}\cdot(2n+1)!}[f^{(2n+2)}(a+\varphi_1x) + 2^{2n+1}f(a+\varphi_2x) + \cdots \\
&\quad + m^{2n+1}f^{(2n+2)}(a+\varphi_mx)].
\end{aligned} \tag{3}
$$

* Mr. Corey gave a partial solution of this problem in the AMERICAN MATHEMATICAL MONTHLY, Vol. 14, Nos. 6–7, pp. 131–135; June–July, 1907.

† The value in final form is given in equation (13) below.

Again from Taylor's development it follows that

$$
(4)\qquad \begin{aligned} f^{(s)}(a+x)-f^{(s)}(a) &= xf^{(s+1)}(a)+\frac{x^2}{2!}f^{(s+2)}(a)+\cdots \\ &+\frac{x^{2n-s+1}}{(2n-s+1)!}f^{(2n+1)}(a)+\frac{x^{2n-s+2}}{(2n-s+2)!}f^{(2n+2)}(a+\theta_s x), \end{aligned}
$$

where $0<\theta_s<1$. We shall again form a series of equations and take their sum. Let $s=1$ and multiply the resulting equation by $-x/m\cdot 2$. Then take $s=2$ and multiply the resulting equation by $-B_1x^2/m^2\cdot 2!$. Then take $s=4$ and multiply by $B_2x^4/m^4\cdot 4!$, and so on. By addition we should finally have the following:

$$
\begin{aligned}
&-\frac{x}{m\cdot 2}[f'(a+x)-f'(a)]-\frac{B_1x^2}{m^2\cdot 2!}[f''(a+x)-f''(a)]+\frac{B_2x^4}{m^4\cdot 4!} \\
&\times[f^{(\text{iv})}(a+x)-f^{(\text{iv})}(a)]-\cdots+(-1)^n\frac{B_nx^{2n}}{m^{2n}\cdot(2n)!}[f^{(2n)}(a+x)-f^{(2n)}(a)]
\end{aligned}
$$

$$
(5)\qquad \begin{aligned}
&=x^2f''(a)\left[-\frac{1}{m\cdot 2}\right]+x^3f'''(a)\left[-\frac{1}{2m\cdot 2!}-\frac{B_1}{m^2\cdot 2!}\right]+x^4f^{(\text{iv})}(a) \\
&\times\left[-\frac{1}{2m\cdot 3!}-\frac{B_1}{m^2\cdot 2!\cdot 2!}\right]+x^5f^{(\text{v})}(a)\left[-\frac{1}{2m\cdot 4!}-\frac{B_1}{m^2\cdot 2!\cdot 3!}+\frac{B_2}{m^4\cdot 4!}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+x^{2n+2}\left[-\frac{1}{2m\cdot(2n+1)!}f^{(2n+2)}(a+\theta_1x)-\frac{B_1}{m^2\cdot 2!(2n)!}f^{(2n+2)}(a+\theta_2x)\right. \\
&\left.+\frac{B_2}{m^4\cdot 4!(2n-2)!}f^{(2n+2)}(a+\theta_4x)-\cdots+(-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\cdot 2!}f^{(2n+2)}(a+\theta_{2n}x)\right].
\end{aligned}
$$

If now we add to the sum of the first members of (3) and (5) the terms $f(a)$ and R_{2n+2} we reproduce the second member of (1). Hence by (1),

$$
f(a+x)=f(a)+\text{second member of (3)}+\text{second member of (5)}+R_{2n+2}.
$$

In order to simplify the sum of these second members, let us consider the coefficients of the two expressions $x^{2r}f^{(2r)}(a)$ and $x^{2r+1}f^{(2r+1)}(a)$. In the sum of the second members of (3) and (5) the coefficient of $x^{2r}f^{(2r)}(a)$ is

$$
(6)\qquad \begin{aligned}
&\frac{1^{2r-1}+2^{2r-1}+\cdots+m^{2r-1}}{m^{2r}\cdot(2r-1)!}-\frac{1}{2m\cdot(2r-1)!}-\frac{B_1}{m^2\cdot 2!(2r-2)!} \\
&\qquad+\frac{B_2}{m^4\cdot 4!(2r-4)!}-\cdots+(-1)^{r-1}\frac{B_{r-1}}{m^{2r-2}\cdot 2!(2r-2)!}.
\end{aligned}
$$

The coefficient of $x^{2r+1}f^{(2r+1)}(a)$ is

$$\frac{1^{2r}+2^{2r}+\cdots+m^{2r}}{m^{2r+1}\cdot(2r)!}-\frac{1}{2m\cdot(2r)!}-\frac{B_1}{m^2\cdot 2!(2r-1)!}+\frac{B_2}{m^4\cdot 4!(2r-3)!} \tag{7}$$
$$-\cdots+(-1)^{r-1}\frac{B_{r-1}}{m^{2r-2}\cdot(2r-2)!2!}+(-1)^r\frac{B_r}{m^{2r}\cdot(2r)!}.$$

But by a theorem due to Bernouilli we have

$$1^r+2^r+3^r+\cdots+m^r=\frac{m^{r+1}}{r+1}+\frac{1}{2}m^r+\frac{r!}{(r-1)!2!}B_1m^{r-1} \tag{8}$$
$$-\frac{r!}{(r-3)!4!}B_2m^{r-3}+\frac{r!}{(r-5)!6!}B_3m^{r-5}\cdots,$$

the second member ending with

$$(-1)^{\frac{r-2}{2}}\frac{r!}{1!\,r!}B_{\frac{r}{2}}m,$$

if r is even and with

$$(-1)^{\frac{r-3}{2}}\frac{r!}{2!\,(r-1)!}B_{\frac{r-1}{2}}m^2,$$

if r is odd. (See Chrystal's Algebra, vol. II, page 209 (2).)

Replacing r by $2r-1$ and substituting in (6) we have for the coefficient of $x^{2r}f^{(2r)}(a)$ an expression which readily reduces to $1/(2r)!$. If we replace r by $2r$ and substitute in (7) we find that the coefficient of $x^{2r+1}f^{(2r+1)}(a)$ reduces to $1/(2r+1)!$. Hence the relation $f(a+x)=f(a)+$ second member of (3) $+$ second member of (5) $+R_{2n+2}$ will reduce to

$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{2!}f''(a)+\cdots+\frac{x^{2n+1}}{(2n+1)!}f^{(2n+1)}(a)$$
$$+\frac{x^{2n+2}}{m^{2n+2}\cdot(2n+2)!}[f^{(2n+2)}(a+\varphi_1x)+2^{2n+1}f^{(2n+2)}(a+\varphi_2x)+\cdots$$
$$+m^{2n+1}f^{(2n+2)}(a+\varphi_mx)]+x^{2n+2}\Bigg[-\frac{1}{2^m\cdot(2n+1)!}f^{(2n+2)}(a+\theta_1x) \tag{9}$$
$$-\frac{B_1}{m^2\cdot 2!\,(2n)!}f^{(2n+2)}(a+\theta_2x)+\frac{B_2}{m^4\cdot 4!\,(2n-2)!}f^{(2n+2)}(a+\theta_4x)$$
$$-\cdots+(-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\,2!}f^{(2n+2)}(a+\theta_{2n}x)\Bigg]+R_{2n+2}.$$

Equation (9) is identical with Taylor's development up to the term containing x^{2n+1}; therefore the remainder after this term in (9) may be equated to the well-known remainder

$$\frac{x^{2n+2}}{(2n+2)!}f^{(2n+2)}(a+\theta x),\qquad 0<\theta<1,$$

after the corresponding term in Taylor's series. This readily yields the value of R_{2n+2} which is sought:

$$
\begin{aligned}
R_{2n+2} = {} & \frac{x^{2n+2}}{(2n+2)!} f^{(2n+2)}(a+\theta x) - \frac{x^{2n+2}}{m^{2n+2}\cdot(2n+2)!}[f^{(2n+2)}(a+\varphi_1 x) \\
& + 2^{2n+1} f^{(2n+2)}(a+\varphi_2 x) + \cdots + m^{2n+1} f^{(2n+2)}(a+\varphi_m x)] \\
& + x^{2n+2}\left[\frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\theta_1 x) + \frac{B_1}{m^2\cdot 2!\,(2n)!} f^{(2n+2)}(a+\theta_2 x)\right. \\
& - \frac{B_2}{m^4\cdot 4!\,(2n-2)!} f^{(2n+2)}(a+\theta_4 x) + \cdots \\
& \left. - (-1)^n \frac{B_n}{m^{2n}\cdot(2n)!\,2!} f^{(2n+2)}(a+\theta_{2n} x)\right].
\end{aligned} \tag{10}
$$

In order to simplify this result we shall make use of the following lemma:

If $a_1, a_2, \cdots, a_i$ are all positive and $\varphi_1, \varphi_2, \cdots, \varphi_i$ are all positive and less than unity, then there exists a positive φ, also less than unity, such that

$$
\begin{aligned}
& a_1 F(a+\varphi_1 x) + a_2 F(a+\varphi_2 x) + \cdots + a_i F(a+\varphi_i x) \\
& \qquad = (a_1 + a_2 + \cdots + a_i) F(a+\varphi x),
\end{aligned} \tag{11}
$$

provided that the function F is continuous.

This may be proved thus: If A and B are the greatest and least values of F in the first member of (11), then is this member increased by replacing F by A and decreased by replacing F by B. Hence, some value, say $F(a+\varphi x)$, between these limits may replace each F in (11) without altering the value of the expression. Hence the theorem. Evidently the proposition also holds if all the a's are negative.

Applying this lemma to equation (10), we may replace each $f^{(2n+2)}$ in the first brackets by $f^{(2n+2)}(a+\varphi x)$. The coefficient of $x^{2n+2} f^{(2n+2)}(a+\varphi x)$ then is

$$
-\frac{1}{m^{2n+2}\cdot(2n+2)!}(1 + 2^{2n+1} + 3^{2n+1} + \cdots + m^{2n+1}).
$$

Changing the form of this coefficient by (8) and substituting in (10) we have

$$
\begin{aligned}
R_{2n+2} = x^{2n+2} & \left[\frac{1}{(2n+2)!} f^{(2n+2)}(a+\theta x) - \frac{1}{(2n+2)!} f^{(2n+2)}(a+\varphi x)\right. \\
& - \frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\varphi x) - \frac{B_1}{m^2\cdot 2!(2n)!} f^{(2n+2)}(a+\varphi x) + \cdots \\
& + (-1)^n \frac{B_n}{m^{2n}\cdot(2n)!\,2!} f^{(2n+2)}(a+\varphi x) + \frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\theta_1 x)
\end{aligned} \tag{12}
$$

$$+\frac{B_1}{m^2\cdot 2!(2n)!}f^{(2n+2)}(a+\theta_2 x)-\frac{B_2}{m^4\cdot 4!(2n-2)!}f^{(2n+2)}(a+\theta_4 x)+\cdots$$

$$-(-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\,2!}f^{(2n+2)}(a+\theta_{2n}x)\Bigg].$$

It will be noticed that this remainder may have its terms grouped into two sets whose terms correspond, term of one to term of the other, and such that all the terms of one set have positive, and all those of the other have negative, coefficients. Then by the lemma already proved, the f's in the two sets may be replaced respectively by $f^{(2n+2)}(a+\varphi_1 x)$ and $f^{(2n+2)}(a+\varphi_2 x)$ where φ_1 and φ_2 are positive and less than unity. Making these changes in (12) we have the following final value for the remainder term:

$$
\begin{aligned}
R_{2n+2} = x^{2n+2}\Bigg[&\frac{1}{(2n+2)!}+\frac{1}{2m\cdot(2n+1)!}+\frac{B_1}{m^2\cdot 2!(2n)!}+\frac{B_2}{m^4\cdot 4!(2n-2)!} \\
&+\cdots+\frac{B_{n-1}}{m^{2n-2}\cdot(2n-2)!\,4!}+\frac{B_n}{m^{2n}\cdot(2n)!\,2!}\Bigg] \\
&\times[f^{(2n+2)}(a+\varphi_1 x)-f^{(2n+2)}(a+\varphi_2 x)].
\end{aligned}
\tag{13}
$$

NEW BOOKS.

Under this heading will be listed all new books received, which pertain to the general subject of mathematics as related to the collegiate and advanced secondary fields. So far as possible either short descriptive notices or more extended reviews will be given, according to the judgment of the committee having this department in charge. The chairman of this committee is Professor W. H. Bussey, of the University of Minnesota, Minneapolis, Minn., and the other members are Professor C. H. Ashton, University of Kansas, Professor W. C. Brenke, University of Nebraska, and Professor L. C. Karpinski, University of Michigan.

Publishers desiring to use this means of conveying to teachers of mathematics information concerning new books will please forward such books to the chairman of the committee.